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THE WAVELIKE MOTION OF A MULTILINK SYSTEM ON A HORIZONTAL PLANE†

F. L. CHERNOUS'KO

Moscow

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The motion of a plane multilink system on a horizontal plane is investigated assuming the existence of dry friction. The motion occurs, under the action of internal control torques applied at the system's joints. It is shown that modes of slow (quasi-static) motion exist in which the system advances along itself owing to a wave travelling along it, in which several links participate. These modes of motion differ from previously studied fast (dynamical) modes of motion of multilink systems along a plane.

1. INTRODUCTION

The majority of known modes of motion of living organisms and devices along a surface have the property that the points of contact of the moving body with the surface do not remain unchanged. Thus, in walking and running the supporting legs alternate; in the motion of wheeled and caterpillar machines the points of contact of the wheels and the tracks with the surface change. In this sense, the motion of snakes and other legless animals, which are in permanent contact with the supporting surface over almost all their length, is a special case. These motions are maintained by torques perpendicular to the supporting surface along which the motion occurs. Note that the motion of wheeled and walking devices involves control torques applied along axes parallel to the supporting surface.

The biomechanics of the motion of living organisms, including snakes, has been investigated in [1, 2]. Several papers [3–7] have been devoted to the kinematics of multilink devices imitating the motion of a snake. These mechanisms are non-holonomic systems equipped with wheels.

The motions of a multilink system moving over a rough horizontal plane, under the action of control torques perpendicular to the plane, applied to the joints of the system, have been constructed and investigated [8, 9]. It has been shown that by alternating the slow and fast phases of the motion the system may move along itself, sideways, or rotate on the sport. Sufficient conditions have been established for these motions to be possible, and the displacements and velocities of the system have been estimated.

In this paper too, we will consider the motion of a multilink system along a rough horizontal plane. It will be shown that forward motion of the system can occur, remaining within the framework of slow (quasi-static) motions, without requiring the application of the large control torques necessary for the fast phases of the motion. Two types of slow motion, of a wavelike nature, are constructed. In one of them, three links of the system are involved at each instant of time; in the other, four are involved. The forces and moments for these motions are computed.

2. THE MECHANICAL MODEL

Consider a multilink system consisting of N identical links, assumed to be absolutely rigid straight rods of length a. For simplicity, we will assume that the masses of the links are negligibly small compared with those of the joints, each of which is a point mass m. The end points of the system have the same mass m. The system is lying on a stationary rough horizontal plane, attached to which is a rectangular Cartesian system coordinates $Oxyz$. The Ox and Oy axes lie in the plane and the Oz axis points vertically upwards.

At each joint P_i ($i = 1, \ldots, N-1$) internal control torques may act directed along the Oz axis. We let M_i denote the torque exerted by the link $P_{i-1}P_i$ on the link P_iP_{i+1}; then the torque exerted by P_iP_{i+1} on $P_{i-1}P_i$ will be $-M_i$ ($i = 1, \ldots, N-1$).

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Between each point P_i and the Oxy plane a dry friction force $\mathbf{F}_i$ acts in the plane. By Coulomb's law, we have

$$\mathbf{F}_i = -mgkv_i^{-1}\mathbf{v}_i, \quad v_i \neq 0$$
$$|\mathbf{F}_i| \leqslant mgk, \quad v_i = 0, \quad i = 0, 1, \ldots, N-1 \tag{2.1}$$

where $\mathbf{v}_i$ is the velocity vector of $\mathbf{P}_i$, v_i is the magnitude of the latter, g is the acceleration due to gravity and, k is the constant coefficient of fraction. At rest ($v_i = 0$) the friction force may have an arbitrary direction.

Let $\mathbf{R}_i$ denote the force exerted by the link $P_{i-1}P_i$ on the point P_i. We will write down equilibrium conditions for the rigid body consisting of the point P_i and the weightless link P_iP_{i+1} (see Fig. 1). The forces acting on this body are: the friction force $\mathbf{F}_i$, the force $\mathbf{R}_i$ exerted by the link $P_{i-1}P_i$, the force $-\mathbf{R}$ exerted by the point P_{i+1}, and also the torques $\mathbf{M}_i$ and $-\mathbf{M}_{i+1}$ exerted by the adjacent links. The equations of equilibrium of the body have the form

$$\mathbf{F}_i + \mathbf{R}_i - \mathbf{R}_{i+1} = 0$$
$$\mathbf{M}_i - \mathbf{M}_{i+1} + \mathbf{r}_{i,i+1} \times (-\mathbf{R}_{i+1}) = 0, \quad i = 1, \ldots, N-2 \tag{2.2}$$

Throughout, $\mathbf{r}_{i,j}$ denotes a vector P_iP_j between the relevant point, $i, j = 0, 1, \ldots, N$. Equations (2.2) hold for all the links except the end ones. For the end links.

$$\mathbf{F}_0 - \mathbf{R}_1 = 0, \quad -\mathbf{M}_1 + \mathbf{r}_{0,1} \times (-\mathbf{R}_1) = 0$$
$$\mathbf{F}_{N-1} + \mathbf{R}_{N-1} - \mathbf{R}_N = 0, \quad \mathbf{M}_{N-1} + \mathbf{r}_{N-1,N} \times (-\mathbf{R}_N) = 0 \tag{2.3}$$
$$\mathbf{F}_N + \mathbf{R}_N = 0$$

The last equation of (2.3) is the equilibrium condition for the point P_N.

In what follows we will construct wavelike slow motions of a multilink system in which it moves along itself under quasi-static conditions. In other words, the velocities and accelerations are assumed to be sufficiently small in magnitude, so that at each instant of time the equilibrium conditions (2.2.) and (2.3) hold to a high degree of accuracy. We will first describe the kinematics of wavelike motions involving three or four moving links, then prove that these motions exist in reality and finally compute the forces and torques.

Let us assume that at the starting time the multilink system is aligned along the axis with the points P_i at coordinates $x_i = ai, y_i = 0$ ($i = 0, 1, \ldots, N$). In that case all the friction forces $\mathbf{F}_i$ ($i = 0, 1, \ldots, N$), reactive forces $\mathbf{R}_i$ ($i = 1, 2, \ldots, N$) and torques M_i ($i = 1, \ldots, N$) are zero.

The initial state of the system is illustrated by state a in Figs 2 and 3. In these and the following figures the points P_i ($i = 0, 1, \ldots, N$) are labelled by digits.

3. WAVELIKE MOTION WITH THREE MOVING LINKS

First the point P_0 advances along the x axis and the points P_i, $i \geqslant 2$, remain fixed. The angle α between the x axis and link P_0P_1 (see state b in Fig. 2) varies monotonically from zero to a certain given value α_0. At this initial stage, the motion involves two links P_0P_1 and P_1P_2, which form an isosceles triangle. At the end of this stage the angle at the base of this triangle will be α_0 (see state c in Fig. 2). All points except P_1 lie on the x axis.

At the next stage the moving links are $P_0P_1P_2$ and P_2P_3. The points P_0 and P_i, $i \geqslant 3$; remain stationary,

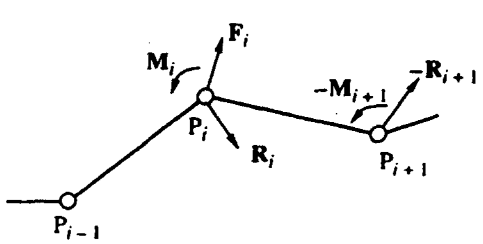

Fig. 1.

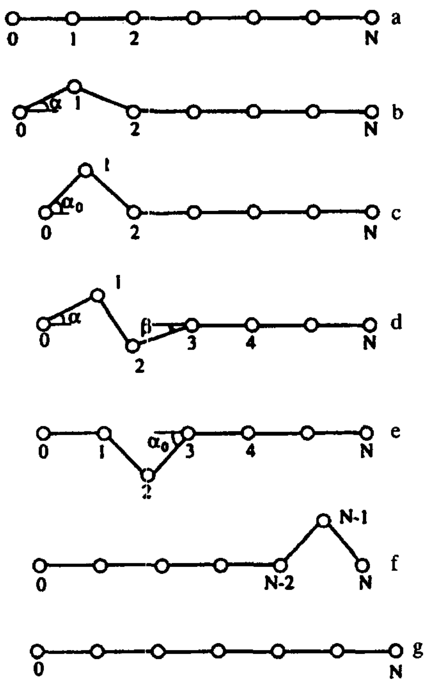

Fig. 2.

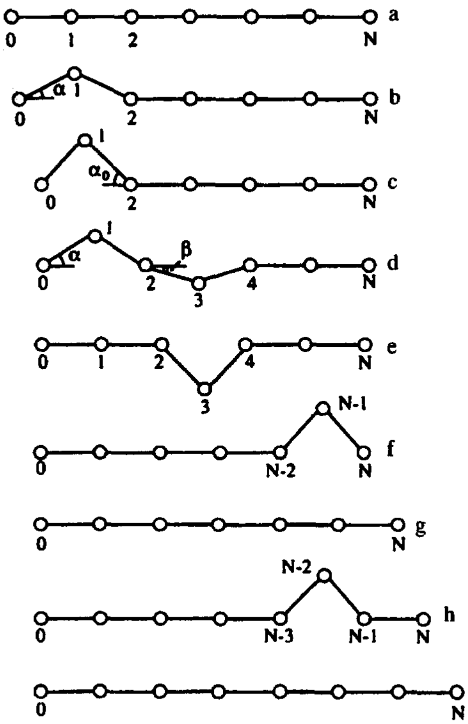

Fig. 3.

and the angle α varies monotonically from α_0 to zero. At the same time the angle β between link P_2P_3 and the x axis varies monotonically from zero to α_0 (see state d in Fig. 2). At the end of this stage the system will be in state e, in which links P_1P_2 and P_2P_3 form an isosceles triangle congruent to the triangle $P_0P_1P_2$ in state c but with its apex pointing in the other direction. Here all points except P_2 lie on the x axis.

Next, the motion will involve links P_1P_2, P_2P_3 and P_3P_4, and this motion, apart from a displacement along the x axis and a mirror reflection in the axis, will be identical with the preceding stage, namely, with the transition from state c to state e. As a result, the point P_2 will return to the x axis, while links P_2P_3 and P_3P_4 form an isosceles triangle congruent with the triangle $P_0P_1P_2$ in state c, a distance $2a$ farther to the right along the x axis.

Continuing this process, we see that after each stage is completed all points of the system except one will lie on the x axis, and that one point will be the apex of an isosceles triangle with angle α_0 at the base. The apex, together with the whole triangle, gradually moves towards the right: it will be the points P_1, P_2, P_3, etc. in turn. Finally, the point P_{N-1} will become the apex of such a triangle, while the points P_i, $0 \leqslant i \leqslant N-2$, and P_N will lie on the x axis (see state f in Fig. 2). The apex of the triangle $P_{N-2}P_{N-1}P_N$, namely, P_{N-1}, will lie on the same side of the x axis as the triangle $P_0P_1P_2$ in state c if N is even (as in Fig. 2), and on the opposite side if N is odd.

In both cases, in the last stage of the motion the point P_N advances to the right along the x axis. The angle α at the base of the triangle $P_{N-2}P_{N-1}P_N$ varies monotonically from α_0 to zero and the multilink system will take up a "straight-line" stage g.

In consequence of the entire cycle of movements, the system will advance along the x axis for a distance L equal to the displacement of the point P_0 from state a to state c in Fig. 2. We have

$$L = 2a(1-\cos\alpha_0) \tag{3.1}$$

4. WAVELIKE MOTION WITH FOUR MOVING LINKS

The first stage of the motion proceeds exactly as in the previous case, and the multilink system goes from state a through an intermediate state b to state c in Fig. 3, which is identical with state c in Fig. 2.

At the next stage of the motion, links P_0P_1, P_1P_2, P_2P_3 and P_3P_4 are involved. The point P_2 moves to the right along the x axis. As this happens the angle α at the base of the isosceles triangle $P_0P_1P_2$ varies monotonically from α_0 to zero, while the angle β at the base of the isosceles triangle $P_2P_3P_4$ varies monotonically from zero to α_0 (see state d in Fig. 3). The apex P_3 of the latter triangle lies on the side of the x axis apposite to that of P_1. At the end of this stage, all points of the system except P_3 will lie on the x axis; P_3 will be the apex of an isosceles triangle with angle α_0 at the base (see state *e* in Fig. 3).

Now links P_2P_3, P_3P_4, P_4P_5 and P_5P_6 take part in the motion. The motion is analogous to that in the preceding state, but displaced along the x axis and reflected in that axis. The result is an isosceles triangle with apex P_5 and angle α_0 at the base.

Continuing the process, we see that the apices of the triangles at the end of each stage are the points with odd indices. Therefore, if N is even, the system will finally reach a state f analogous to state f in Fig. 2. In that case, if $N/2$ is odd, the apex P_{N-1} of the triangle $P_{N-2}P_{N-1}P_N$ will point in the same direction as the apex P_1 of the triangle $P_0P_1P_2$ in the initial stage (see Fig. 3, f); if $N/2$ is even, it will point in the opposite direction. In both cases, in the last stage the point P_N will move to the right along the x axis until the entire system takes up the straight position g shown in Fig. 3.

On the other hand, if N is odd, one obtains situation h of Fig. 3. All points except P_{N-2} will lie on the x axis, and P_{N-2} will be the apex of an isosceles triangle with angle α_0 at the base. This vertex will point in the same direction as the apex P_1 of the triangle $P_0P_1P_2$ in state c if $(N-1)/2$ is odd (this is the case shown in Fig. 3), and in the opposite direction if $(N-1)/2$ is even. In both cases, the last stage is that the points P_{N-1} and P_N will move to the right along the x-axis, until the system takes up the straight position i shown in Fig. 3.

The total advance of the system along the x axis through the whole cycle is determined by the same relation (3.1) as in the previous case.

5. THE QUASI-STATIC APPROACH

As already pointed out, we are considering motions of a multilink system in a quasi-static formulation, with velocities and accelerations assumed to be extremely small. In this formulation all external forces applied to the system must almost balance out. Such external forces acting in the plane of the motion are friction forces. In a first approximation, therefore, we must require the friction forces to satisfy three equilibrium conditions (two for the forces and one for the moments). The friction forces applied to the moving points are readily evaluated if we know the directions of the velocities (see (2.1)). For points at rest, however, the friction forces are unknown, but they are bounded by inequalities. The system is statically indeterminate if the number of unknown components of the friction forces exceeds three. Consequently, the equilibrium problem need not necessarily have a unique solution.

We will try to find the simplest distributions of the friction forces for which equilibrium is attained with the participation of the least possible number of stationary points adjacent to the moving points. Points P_i at which the friction force is not zero will be called active. We are thus looking for a solution of the problem of statics with the least possible number of active points adjacent to moving points.

As follows from the description of the wavelike motions of the system, these motions consist of the following distinct stages.

A. The initial stage, that is, transition from state a to state c through an intermediate state b as in Figs 2 and 3.

B. The final stage, that is, transition from state e to state g in Figs 2 and 3; this stage is similar to the initial stage; it always takes place in motion with three moving links, and if N is even – also in motion with four moving links.

C. The final stage in motion with four moving links in the case of odd N, corresponding to transition from state h to state i in Fig. 3.

D. The basic stage of motion with three moving links, characterized by state d in Fig. 2.

E. The basic stage of motion with four moving links, characterized by state d in Fig. 3.

We will now consider equilibrium conditions for each of stages A–E.

6. THE INITIAL STAGE

Consider the intermediate state b of initial stage A of the motion (Figs 2 and 3). The coordinates of the moving points P_0 and P_1 are, respectively

$$x_0 = 2a - 2a\cos\alpha,\quad y_0 = 0$$
$$x_1 = 2a - a\cos\alpha,\quad y_1 = a\sin\alpha \tag{6.1}$$

and their velocities have components

$$\dot{x}_0 = 2a\dot{\alpha}\sin\alpha,\quad \dot{y}_0 = 0$$
$$\dot{x}_1 = a\dot{\alpha}\sin\alpha,\quad \dot{y}_1 = a\dot{\alpha}\cos\alpha \tag{6.2}$$

To fix our ideas, let us assume that $\alpha \in (0, \pi/2)$, and then $\dot{\alpha} > 0$. The friction forces acting at the points P_0 and P_1 are defined by (2.1) in which the velocities are substituted from (6.2). We have

$$\mathbf{F}_0 = -mgk\mathbf{i},\quad \mathbf{F}_1 = -mgk(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}) \tag{6.3}$$

Henceforth, we will let **i** and **j** denote unit vectors along the coordinate axes Ox and Oy, respectively. The projections of the friction forces $\mathbf{F}_i$ onto the axes Ox and Oy will be denoted by Xi and Yi, respectively.

Let us assume that the active points, that is, those with non-zero friction forces, are P_i, $i \leqslant 5$. We set up the equilibrium equations for the system, considering these points only and taking relations (6.3) into account:

$$X_2 + X_3 + X_4 + X_5 = mgk(1 + \sin\alpha) \tag{6.4}$$

$$Y_2 + Y_3 + Y_4 + Y_5 = mgk\cos\alpha \tag{6.5}$$

$$\mathbf{r}_{2,1} \times \mathbf{F}_1 + a\mathbf{i} \times \mathbf{F}_3 + 2a\mathbf{i} \times \mathbf{F}_4 + 3a\mathbf{i} \times \mathbf{F}_5 = 0 \tag{6.6}$$

Equation (6.6) is the equation of moments about the point P_2. Substituting the components ($-a\cos\alpha$, $a\sin\alpha$ of the vector $\mathbf{r}_{2,1}$ and the force $\mathbf{F}_1$ from (6.3) into this equation, we obtain

$$Y_3 + 2Y_4 + 3Y_5 = -mgk \tag{6.7}$$

In order to satisfy the equilibrium conditions we have to find a solution of system (6.4), (6.5), (6.7) satisfying conditions (2.1):

$$X_i^2 + Y_i^2 \leqslant (mgk)^2 \tag{6.8}$$

In what follows it will be convenient to normalize all the components of the friction forces. We set

$$X_i = mgk\xi_i,\quad Y_i = mgk\eta_i,\quad i = 0, 1, \ldots, N \tag{6.9}$$

Then the restrictions of Coulomb's law (2.1) (or (6.8)) become

$$\xi_i^2 + \eta_i^2 \leqslant 1,\quad i = 0, 1, \ldots, N \tag{6.10}$$

The equilibrium equations (6.4), (6.5) and (6.7) may now be rewritten in the form

$$\begin{aligned} &\xi_2 + \xi_3 + \xi_4 + \xi_5 = 1 + \sin\alpha \\ &\eta_2 + \eta_3 + \eta_4 + \eta_5 = \cos\alpha \\ &\eta_3 + 2\eta_4 + 3\eta_5 = -1 \end{aligned} \tag{6.11}$$

We will first try to satisfy Eqs (6.11) with two fixed active points P_2 and P_3, setting $\xi_4 = \eta_4 = \xi_5 = \eta_5 = 0$. Then the third equation of (6.11) yields $\eta_3 = -1$. Consequently, by (6.10), $\xi_3 = 0$. The first equation of (6.11) then gives $\xi_2 = 1 + \sin\alpha$, which violates condition (6.10).

Thus, two fixed active points are not sufficient, and the point P_4 must also be active. Put $\xi_5 = \eta_5 = 0$. Multiply the second equation of (6.11) by two and subtract the third equation of (6.11) from the result. We obtain

$$2\eta_2 + \eta_3 = 1 + 2\cos\alpha \tag{6.12}$$

At the beginning of the initial stage we have $\alpha = 0$, and Eq. (6.12) gives $2\eta_2 + \eta_3 = 3$. It follows from inequalities (6.10) that $\eta_2 = \eta_3 = 1$. Substituting $\eta_3 = 1$ and $\eta_5 = 0$ into the third equation, we obtain $\eta_4 = -1$. Thus, for $i = 2, 3, 4$ we have $|\eta_i| = 1$ and, by (6.10), $\xi_i = 0$. Hence the first equation of (6.11) is not satisfied when $\xi_5 = 0$.

Consequently, all four points P_2, P_3, P_4, P_5 must be active. The two possible solutions of Eqs (6.11) and inequalities (6.10) are

$$\begin{aligned} &\xi_2 = \xi_4 = 0, \quad \xi_3 = \sin\alpha, \quad \xi_5 = 1 \\ &\eta_2 = -\eta_4 = (1+\cos\alpha)/2, \quad \eta_3 = \cos\alpha, \quad \eta_5 = 0 \end{aligned} \tag{6.13}$$

$$\begin{aligned} &\xi_2 = \xi_5 = \tfrac{1}{2}, \quad \xi_3 + \xi_4 = (\sin\alpha)/2 \\ &\eta_2 = \tfrac{1}{3} + (\cos\alpha)/2, \quad \eta_3 = \cos\alpha, \quad \eta_4 = -(\cos\alpha)/2, \quad \eta_5 = -\tfrac{1}{3} \end{aligned} \tag{6.14}$$

A direct check will show that both solutions (6.13) and (6.14) satisfy Eqs (6.11) and inequalities (6.10). In actual fact, for (6.13) some of the inequalities become equalities, but for (6.14) all the inequalities (6.10) hold "with room to spare".

7. THE FINAL STAGES

The final stage B, characterized by the transition from state f to state g in Figs 2 and 3, is essentially the initial stage in retrograde time. The points P_{N-1} of stage B should be identified with the points P_i in stage A ($i = 0, 1, 2, 3, 4, 5$). The sign of the angular velocity $\dot{\alpha}$ is reversed, and the point P_N of stage B, like P_0 in stage A, advances along the x axis. As a result it turns out that the x-components of all three friction forces have the same signs as before, but the y-components change sign. We have the following correspondence of forces for stages B and A

$$X_{N-i} = X_i, \quad Y_{N-i} = -Y_i, \quad i = 0, 1, 2, 3, 4, 5 \tag{7.1}$$

The proof of the previous section that four fixed points must be active, and the derivation of the solution (6.13), (6.14), remain valid, provided one takes the notation (7.1) and (6.9) into account.

We now consider the final stage C, which takes place in the case of four moving points when $(N-1)/2$ is even (situation h in Fig. 3). At this stage (see Fig. 4) the points P_i, $i \leqslant N-3$, remain stationary, while P_{N-1} and P_N move along the x axis. We place the origin at the point P_{N-3}, direct the x axis along the link $P_{N-4}P_{N-3}$ and write down the coordinates and velocities of the moving points

$$\begin{aligned} &x_{N-2} = a\cos\alpha, \quad y_{N-2} = a\sin\alpha, \quad x_{N-1} = 2a\cos\alpha \\ &y_{N-1} = 0, \quad x_N = 2a\cos\alpha + a, \quad y_N = 0 \\ &\dot{x}_{N-2} = -a\dot{\alpha}\sin\alpha, \quad \dot{y}_{N-2} = a\dot{\alpha}\cos\alpha \\ &\dot{x}_{N-1} = \dot{x}_N = -2a\dot{\alpha}\sin\alpha, \quad \dot{y}_{N-1} = \dot{y}_N = 0 \end{aligned} \tag{7.2}$$

where α is the angle at the base of the isosceles triangle $P_{N-3}P_{N-2}P_{N-1}$, $\dot{\alpha} < 0$.

Using Eq (2.1) and relations (7.2), we proceed as was done for (6.3) to evaluate the friction forces acting on the moving points

$$\mathbf{F}_{N-2} = mgk(-\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}), \quad \mathbf{F}_{N-1} = \mathbf{F}_N = -mgk\mathbf{i} \tag{7.3}$$

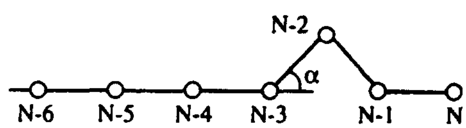

Fig. 4.

The equilibrium equations considering the points from P_{N-6} to P_N, are

$$\begin{aligned} &X_{N-6}+X_{N-5}+X_{N-4}+X_{N-3}=mgk(2+\sin\alpha)\\ &Y_{N-6}+Y_{N-5}+Y_{N-4}+Y_{N-3}=-mgk\cos\alpha\\ &-3a\mathbf{i}\times\mathbf{F}_{N-6}-2a\mathbf{i}\times\mathbf{F}_{N-5}-a\mathbf{i}\times\mathbf{F}_{N-4}+\mathbf{r}_{N-3,N-2}\times\mathbf{F}_{N-2}+\\ &+\mathbf{r}_{N-3,N-1}\times\mathbf{F}_{N-1}+\mathbf{r}_{N-3,N}\times\mathbf{F}_N=0 \end{aligned} \tag{7.4}$$

In the equation of moments (7.4) about the point P_{N-3} we substitute the friction forces (7.3) and the coordinates of the points P_{N-2}, P_{N-1}, P_N from (7.2). As a consequence, taking the normalization (6.9) into account, the system of equilibrium equations (7.4) becomes

$$\begin{aligned} &\xi_{N-6}+\xi_{N-5}+\xi_{N-4}+\xi_{N-3}=2+\sin\alpha\\ &\eta_{N-6}+\eta_{N-5}+\eta_{N-4}+\eta_{N-3}=-\cos\alpha\\ &3\eta_{N-6}+2\eta_{N-5}+\eta_{N-4}=1 \end{aligned} \tag{7.5}$$

Let us assume that, in all, there are three active fixed points, P_{N-3}, P_{N-4} and P_{N-5}, so that $\zeta_{N-6}=\eta_{N-6}=0$. Multiply the second equation of (7.5) by two and subtract the third equation of (7.5) from the result. This gives

$$\eta_{N-4}+2\eta_{N-3}=-2\cos\alpha-1 \tag{7.6}$$

If $\alpha=0$, it follows from (7.6) that $\eta_{N-4}+2\eta_{N-3}=-3$, but since $|\eta_i|\leq 1$ for all i, it follows that $\eta_{N-4}=\eta_{N-3}=-1$. By (6.10), we have $\xi_{N-4}=\xi_{N-3}=0$. Then the first equation of (7.5), given that $\xi_{N-6}=0$, $\alpha=0$, becomes $\xi_{N-5}=2$, and inequality (6.10) is not satisfied for $i=N-5$. Consequently, three active fixed points are insufficient, and the point P_{N-6} is also active.

One possible solution of Eqs (7.5), also satisfying inequalities (6.10) (with "room to spare"), is the following

$$\begin{aligned} &\xi_{N-6}=\tfrac{5}{6},\quad \xi_{N-5}=(4+\sin\alpha)/6,\quad \xi_{N-4}=(5\sin\alpha)/6\\ &\xi_{N-3}=\tfrac{1}{2},\quad \eta_{N-6}=\tfrac{1}{3},\quad \eta_{N-5}=(\cos\alpha)/2\\ &\eta_{N-4}=-\cos\alpha,\quad \eta_{N-3}=-(2+3\cos\alpha)/6 \end{aligned}$$

8. THE BASIC STAGE FOR THREE MOVING LINKS

This stage (stage D in Section 6) is characterized by state d in Fig. 2. The general case of this stage is illustrated in Fig. 5, where the moving links are P_iP_{i+1}, $P_{i+1}P_{i+2}$ and $P_{i+2}P_{i+3}$. We place the origin at the stationary point P_i, direct the x axis along the segment P_iP_{i+3} and write down the coordinates of the moving points P_{i+1} and P_{i+2}

$$\begin{aligned} &x_{i+1}=a\cos\alpha,\quad y_{i+1}=a\sin\alpha\\ &x_{i+2}=l-a\cos\beta,\quad y_{i+2}=-a\sin\beta \end{aligned} \tag{8.1}$$

where α and β are the angles between the x axis and the moving links P_iP_{i+1} and $P_{i+2}P_{i+3}$, measured in opposite directions (see Fig. 5), and l is the distance P_iP_{i+3}, which remains unchanged throughout this stage.

Differentiating relations (8.1), we determine the velocities of the points P_{i+1} and P_{i+2}

$$\begin{aligned} &\dot{x}_{i+1}=-a\dot{\alpha}\sin\alpha,\quad \dot{y}_{i+1}=a\dot{\alpha}\cos\alpha\\ &\dot{x}_{i+2}=a\dot{\beta}\sin\beta,\quad \dot{y}_{i+2}=-a\dot{\beta}\cos\beta \end{aligned} \tag{8.2}$$

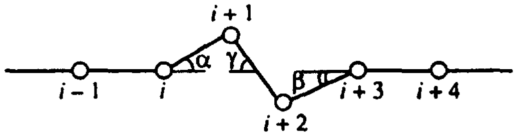

Fig. 5.

Throughout this stage we have $\dot{\alpha} < 0$, $\dot{\beta} > 0$. The friction forces acting on the moving points P_{i+1} and P_{i+2} are determined using relations (2.1) and formulae (8.2). We obtain

$$\mathbf{F}_{i+1} = mgk(-\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j})$$
$$\mathbf{F}_{i+2} = mgk(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \tag{8.3}$$

We set up the equilibrium equations, assuming that there are just two active stationary points on each side of the moving links, that is, the active points are P_j, $i-1 \leqslant j \leqslant i+4$ (see Fig. 5). Using Eqs (8.3), we have

$$\begin{aligned} &X_{i-1} + X_i + X_{i+3} + X_{i+4} = mgk(\sin\alpha + \sin\beta) \\ &Y_{i-1} + Y_i + Y_{i+3} + Y_{i+4} = -mgk(\cos\alpha + \cos\beta) \\ &-aY_{i-1} + lY_{i+3} + (l+a)Y_{i+4} = \\ &= -mgka\begin{vmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{vmatrix} - mgk\begin{vmatrix} l - a\cos\beta & -a\sin\beta \\ -\sin\beta & \cos\beta\end{vmatrix}\end{aligned} \tag{8.4}$$

The last equation of (8.4) is the equation of moments about the point P_i, obtained using (8.1) and (8.3). After simplifying and changing to the dimensionless variables (6.9), Eqs (8.4) become

$$\begin{aligned} &\xi_{i-1} + \xi_i + \xi_{i+3} + \xi_{i+4} = \sin\alpha + \sin\beta \\ &\eta_{i-1} + \eta_i + \eta_{i+3} + \eta_{i+4} = -(\cos\alpha + \cos\beta) \\ &-a\eta_{i-1} + l\eta_{i+3} + (l+a)\eta_{i+4} = -l\cos\beta \end{aligned} \tag{8.5}$$

System (8.5) has a solution

$$\begin{aligned} &\xi_{i-1} = \xi_{i+4} = \eta_{i-1} = \eta_{i-4} = 0 \\ &\xi_i = \sin\alpha, \quad \xi_{i+3} = \sin\beta, \quad \eta_i = -\cos\alpha, \quad \eta_{i+3} = -\cos\beta \end{aligned} \tag{8.6}$$

which also satisfies inequalities (6.10). This solution is interesting because it involves only two active stationary points P_i and P_{i+3}, which are the ends of the moving links.

9. THE BASIC STAGE FOR FOUR MOVING LINKS

This stage (stage E in Section 6) is characterized by state d in Fig. 3. The general case of this stage is illustrated in Fig. 6, where the moving links are P_iP_{i+1}, $P_{i+1}P_{i+2}$, $P_{i+2}P_{1+3}$ and $P_{i+3}P_{i+4}$. We place the origin at the stationary point P_i and direct the x axis along the segment P_iP_{i+4}. Let α and β denote the angles at the bases of the isosceles triangles $P_iP_{i+1}P_{i+2}$ and $P_{i+2}P_{i+3}P_{i+4}$, respectively. The coordinates of the moving points are

$$\begin{aligned} &x_{i+1} = a\cos\alpha, \quad y_{i+1} = a\sin\alpha, \quad x_{i+2} = 2a\cos\alpha \\ &y_{i+2} = 0, \quad x_{i+3} = a(2\cos\alpha + \cos\beta), \quad y_{i+3} = -a\sin\beta \end{aligned} \tag{9.1}$$

and their velocities are

$$\begin{aligned} &\dot{x}_{i+1} = -a\dot{\alpha}\sin\alpha, \quad \dot{y}_{i+1} = a\dot{\alpha}\cos\alpha, \quad \dot{x}_{i+2} = -2a\dot{\alpha}\sin\alpha \\ &\dot{y}_{i+2} = 0, \quad \dot{x}_{i+3} = -a(2\dot{\alpha}\sin\alpha + \dot{\beta}\sin\beta), \quad \dot{y}_{i+3} = -a\dot{\beta}\cos\beta \end{aligned} \tag{9.2}$$

Throughout the motion, the distance P_iP_{i+4} remains unchanged and is equal to its value at the beginning and at the end of the step, when one of the angles α or β is zero and the other has its maximum value α_0. We therefore have

$$\cos\alpha + \cos\beta = 1 + \cos\alpha_0 \tag{9.3}$$

Differentiating this identity, we obtain

$$\dot{\alpha}\sin\alpha + \dot{\beta}\sin\beta = 0 \tag{9.4}$$

In view of (9.4), we deduce from (9.2) that

$$\dot{x}_{i+3} = a\dot{\beta}\sin\beta \tag{9.5}$$

Taking into account that $\dot{\alpha} < 0$, $\dot{\beta} > 0$, we use relations (2.1), (9.2) and (9.5) to determine the friction forces acting on the moving points

$$\begin{aligned} &\mathbf{F}_{i+1} = mgk(-\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}), \quad \mathbf{F}_{i+2} = -mgk\mathbf{i} \\ &\mathbf{F}_{i+3} = mgk(-\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) \end{aligned} \tag{9.6}$$

We set up the equilibrium equations of the multilink system, assuming that on each side of the moving links there are just two stationary points, that is, the active points are P_j, $i - 1 \leqslant j \leqslant i + 5$ (see Fig. 6). Taking Eqs (9.6) into account, we obtain

$$\begin{aligned} &X_{i-1} + X_i + X_{i+4} + X_{i+5} = mgk(1 + \sin\alpha + \sin\beta) \\ &Y_{i-1} + Y_i + Y_{i+4} + Y_{i+5} = -mgk(\cos\alpha + \cos\beta) \\ &-aY_{i-1} + 2a(\cos\alpha + \cos\beta)Y_{i+4} + a(2\cos\alpha + 2\cos\beta + 1)Y_{i+5} = \\ &= -mgka\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} - mgka\begin{vmatrix} 2\cos\alpha + \cos\beta & -\sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix} \end{aligned} \tag{9.7}$$

The last equation of (9.7) is the equation of moments about the point P_i, derived using (9.1) and (9.6).

We simplify Eqs (9.7), using identity (9.3) and changing to dimensionless variables (6.9). We obtain

$$\begin{aligned} &\xi_{i-1} + \xi_i + \xi_{i+4} + \xi_{i+5} = 1 + \sin\alpha + \sin\beta \\ &\eta_{i-1} + \eta_i + \eta_{i+4} + \eta_{i+5} = -(\cos\alpha + \cos\beta) \\ &-\eta_{i-1} + 2(1 + \cos\alpha_0)\eta_{i+4} + (3 + 2\cos\alpha_0)\eta_{i+5} = -2\cos\beta(1 + \cos\alpha_0) \end{aligned} \tag{9.8}$$

As in Section 8, we will first try to limit ourselves to two active stationary points and put $\xi_{i-1} = \eta_{i-1} = \xi_{i+5} = \eta_{i+5} = 0$. It then follows from the third equation of (9.8) that $\eta_{i+4} = -\cos\beta$, and it then follows from the second equation of (9.8) that $\eta_i = -\cos\alpha$. It now follows from inequality (6.10) that $|\xi_i| \leqslant \sin\alpha$, $|\xi_{i+4}| \leqslant \sin\beta$. These inequalities contradict the first equation of (9.8) for $\xi_{i-1} = \xi_{i+5} = 0$. Thus, two active stationary points are insufficient.

It turns out that if at least one of the points P_{i+1} or P_{i+5} is also active, a solution exists. Set

$$\begin{aligned} &\xi_i = \sin\alpha, \quad \eta_i = -\cos\alpha, \quad \xi_{i+4} = \sin\beta, \quad \eta_{i+4} = -\cos\beta \\ &\eta_{i-1} = \eta_{i+5} = 0 \end{aligned} \tag{9.9}$$

and, in addition,

$$\text{either } \xi_{i-1} = 1,\ \xi_{i+5} = 0 \text{ or } \xi_{i-1} = 0,\ \xi_{i+5} = 1 \tag{9.10}$$

It can be readily verified that both solutions (9.9) and (9.10) satisfy Eqs (9.8) and inequalities (6.10). Thus, in this case there must be three active stationary points: P_i, P_{i+4} and either P_{i-1} or P_{i+5}.

10. DETERMINATION THE TORQUES

At any stage of the wavelike motion with three or four moving links, there is a certain finite number of active points. Let P_q be the outermost active point on the left and P_r the outermost one on the right, $q < r$. We have $\mathbf{F}_i \neq 0$ for $q \leqslant i \leqslant r$, $\mathbf{F}_i = 0$ for $i < q$ and $i > r$. At the outermost active points we have

$$\mathbf{R}_q = 0, \quad \mathbf{M}_q = 0; \quad \mathbf{R}_{r+1} = 0, \quad \mathbf{M}_{r+1} = 0 \tag{10.1}$$

Conditions (10.1) mean that there is no force acting on the point P_q and the link P_qP_{q-1} from the left, and none acting on P_r and $P_{r-1}P_r$ from the right.

In Section 2, the forces $\mathbf{R}_i$ are defined for $1 \leqslant i \leqslant N$ and the torques M_i for $1 \leqslant i \leqslant N-1$; this implies that relations (10.1) are meaningful only for $0 < q < r < N-1$. We extend these definitions to $i = 0, N, N+1$, in accordance with their meanings (see Section 2), as follows:

$$\mathbf{R}_0 = \mathbf{R}_{N+1} = 0, \quad \mathbf{M}_0 = \mathbf{M}_N = \mathbf{M}_{N+1} = 0$$

Then, as is easily seen, Eqs (10.1) and the subsequent formulae of Section 10 will hold for all q, r such that $0 \leqslant q < r \leqslant N$.

If the multilink system is in equilibrium or in quasi-static motion, each pair of conditions in (10.1) implies the other pair. Indeed, by (2.2), we have

$$\mathbf{R}_{r+1} = \mathbf{F}_r + \mathbf{R}_r = \mathbf{F}_r + \mathbf{F}_{r-1} + \mathbf{R}_{r-1} = \ldots = \sum_{i=q}^{r} \mathbf{F}_i + \mathbf{R}_q \tag{10.2}$$

In the equilibrium state, all the friction forces on active points are balanced out; hence it follows from (10.2) that the equalities $\mathbf{R}_q = 0$ and $\mathbf{R}_{r+1} = 0$ are equivalent to each other.

Let us compute the torque applied at the i-th joint. Applying Eqs (2.2), we obtain a chain of relations

$$\begin{aligned}
\mathbf{M}_i &= \mathbf{M}_{i-1} + \mathbf{r}_{i,i-1} \times (\mathbf{F}_{i-1} + \mathbf{R}_{i-1}) = \mathbf{r}_{i,i-1} \times \mathbf{F}_{i-1} + \mathbf{M}_{i-2} + \mathbf{r}_{i-1,i-2} \times (\mathbf{F}_{i-2} + \mathbf{R}_{i-2}) + \\
&+ \mathbf{r}_{i,i-1} \times (\mathbf{F}_{i-2} + \mathbf{R}_{i-2}) = \mathbf{r}_{i,i-1} \times \mathbf{F}_{i-1} + \mathbf{r}_{i,i-2} \times \mathbf{F}_{i-2} + \mathbf{M}_{i-2} + \mathbf{r}_{i,i-2} \times \mathbf{R}_{i-2} = \\
&= \ldots = \sum_{j=q}^{i-1} \mathbf{r}_{i,j} \times \mathbf{F}_j + \mathbf{M}_q + \mathbf{r}_{i,q} \times \mathbf{R}_q
\end{aligned} \tag{10.3}$$

Since $\mathbf{R}_q = 0$, $\mathbf{M}_q = 0$, we derive from (10.3) a formula for the torque at the ith joint

$$\mathbf{M}_i = \sum_{j=q}^{i-1} r_{i,j} \times \mathbf{F}_j, \quad q \leqslant i \leqslant r+1 \tag{10.4}$$

Putting $i = r + 1$ in (10.4), and allowing for the fact that the total torque of the friction forces applied to the system at all active points is zero, we obtain $\mathbf{M}_{r+1} = 0$.

Thus, if one pair of conditions (10.1) is satisfied, so is the other.

Formula (10.4) enables us to estimate the torque that must be developed by motors mounted at the joints of the multilink system. Using the solutions for the friction forces obtained in Sections 6–9, we use formula (10.4) to evaluate the torques at the joints for all active points at stages A–E of the motion. It turns out that in all cases, and for all values of the angle α_0,

$$|\mathbf{M}_i| \leqslant 2mgka \tag{10.5}$$

The torques $\mathbf{M}_i$ at the joints, as calculated in this section, guarantee equilibrium of the system in every position. In order to create the desired quasi-static motion according to the scheme of Sections 3 and 4, one has to specify the torques in the form

$$\mathbf{M}_i^* = \mathbf{M}_i + \Delta\mathbf{M}_i \tag{10.6}$$

The torques $\mathbf{M}i$ were determined above, and the additional small torques $\Delta\mathbf{M}_i$ must be formed in accordance with the feedback principle. To that end, the desired law of motion should first be stipulated, e.g. by specifying the generalized coordinates of the system as functions of time: $q_k = q_k^0(t)$ $(k = 1, \ldots, N + 2)$. The functions $q_k^0(t)$ should be such that, as t varies from zero to T, the system successively passes through the states described in Sections 3 and 4. Throughout, the velocities and accelerations must be sufficiently small, so that the inertial forces produced are much smaller than the friction forces. This condition is expressed by the inequalities

$$(\dot{\alpha})^2 a \ll gk, \quad \ddot{\alpha} a \ll gk$$

The additional torques $\Delta\mathbf{M}_i$ should be specified, e.g. as feedback

$$\Delta\mathbf{M}_i = \sum_{k=1}^{N+2} \{\mathbf{W}_{ik}[q_k - q_k^0(t)] + \mathbf{W}_{ik}^*[\dot{q}_k - \dot{q}_k^0(t)]\} \tag{10.7}$$

where the matrices $\mathbf{W}_{ik}$ and $\mathbf{W}^*_{ik}$ must be chosen so that any motion thus realized is stable. The generalized coordinates and velocities q_k, $\dot{q}_k$ should be measured by sensors. It seems that the desired motion may be followed in this way to within a given accuracy. However, this problem requires further investigation.

We merely remark that the additional torques $\Delta\mathbf{M}_i$ can be made very small compared with $\mathbf{M}_i$ by a proper choice of the slow quasi-static motion $q_k^0(t)$. Formulae (10.4) and estimates (10.5), therefore, determine the principal part of the required torques developed by the motors.

11. DISCUSSION OF THE RESULTS

As shown in Sections 6 and 7, at the initial and final stages A–C of wavelike motion there must be at least four active stationary points; at stage D there must be at least two, and at stage E, at least three. Hence wavelike motion of a multilink system with three moving links is feasible in practice if it has at least six points or five links ($N \geqslant 5$). To produce wavelike motion with four moving links, the system must have at least seven points (six links, $N \geqslant 6$). Motion with three moving links is simpler but, as shown below, requires that the angles at which the links are inclined to the x axis be large.

Let us estimate the maximum angles of inclination of the links to the x axis in motion with three and with four moving links, for the same total displacement L of the system. Since formula (3.1) holds for both types of motion, the value of α_0 for the same L will be the same in both types of motion. The angle at which the links are inclined to the x axis in the case of four moving links will not exceed α_0.

Let us determine the maximum angle of inclination of the links to the x axis in the case of motion with three moving links. It can be seen from Fig. 5 that the largest angle of inclination to the x axis is that of the middle link $P_{i+1}P_{i+2}$ of the three moving ones. Letting γ denote the angle between this link and the x axis, we have (see Fig. 5).

$$a(\cos\alpha+\cos\beta+\cos\gamma)=l=a+2a\cos\alpha_0 \tag{11.1}$$

The second equality of (11.1) corresponds to the case in which $\alpha = 0$ or $\beta = 0$ in Fig. 5. The largest inclination of the link $P_{i+1}P_{i+2}$ corresponds to the least value of $\cos\gamma$ or, by (11.1), the maximum value of the sum $\cos\alpha + \cos\beta$. Using Eqs (8.1), we can write down the condition for the length of the link $P_{i+1}P_{i+2}$ to be equal to a. We have

$$[l-a(\cos\alpha+\cos\beta)]^2+a^2(\sin\alpha+\sin\beta)^2=a^2 \tag{11.2}$$

After simplifying, we obtain from (11.2)

$$l(\cos\alpha+\cos\beta)-a\cos(\alpha-\beta)=(l^2+a^2)(2a)^{-1} \tag{11.3}$$

We now determine the conditional extremum of $\cos\alpha + \cos B$, given (11.3). We construct the Lagrange function

$$G=\cos\alpha+\cos\beta+\lambda l(\cos\alpha+\cos\beta)-\lambda a\cos(\alpha-\beta) \tag{11.4}$$

where λ is a Lagrange multiplier, and equate the partial derivatives $\partial G/\partial\alpha$ and $\partial G/\partial\beta$ to zero. We have

$$\begin{aligned} &-(1+\lambda l)\sin\alpha+\lambda a\sin(\alpha-\beta)=0\\ &-(1+\lambda l)\sin\beta-\lambda a\sin(\alpha-\beta)=0 \end{aligned} \tag{11.5}$$

Adding these equations together, we obtain

$$(\sin\alpha+\sin\beta)(1+\lambda l)=0 \tag{11.6}$$

Since the angles α and β are in the interval $[0, \alpha_0]$, where $\alpha_0 < \pi/2$, and they cannot both vanish, it follows from (11.6) that $1 + \lambda l = 0$. We then deduce from (11.5) that $\alpha = \beta$. Substituting this condition into (11.3) and replacing l by its value according to (11.1), we obtain

$$\cos\alpha=\cos\beta=(1+\cos\alpha_0+\cos^2\alpha_0)(1+2\cos\alpha_0)^{-1} \tag{11.7}$$

It can be easily verified that this extremum corresponds to the desired maximum value of $\cos\alpha + \cos\beta$ and lies in the interval $(0, \alpha_0)$.

The corresponding minimum value of $\cos\gamma$ is found from (11.1)

$$\cos\gamma = (2\cos\alpha_0 + 2\cos^2\alpha_0 - 1)(1 + 2\cos\alpha_0)^{-1} \tag{11.8}$$

This following conclusions may be drawn from formulae (11.7) and (11.8). It is always true that $\cos\gamma < \cos\alpha_0$, that is, the angle of inclination in the case of three moving links is larger than in the case of four moving links. If $\alpha_0 > 69°$, that is, $\cos\alpha_0 < (31/2-1)/2$, than $\cos\gamma < 0$, that is, the angle of inclination of the link $P_{i+1}P_{i+2}$ exceeds $\pi/2$. Also if $\alpha_0 \to \alpha/2$, then $\gamma \to \pi$. Thus, in the case of three moving links the multilink system is more strongly bent, especially at large values of α_0.

If the number of links N is sufficiently large, there may be several waves of the above types propagating along the system at the same time. When the first of the waves with three or four moving links has advanced far enough along system, a new wave of the same or of another type may begin at the end of the system (i = 0, 1, 2), advancing along the system after the first. Thus, the average velocity of displacement of the system as a whole may be increased several fold.

To conclude, we note that, simple the modes of motion of a multilink system described previously [8, 9], where implementation of the fast phases of motion required torques considerably exceeding the moments *mgka* of the friction forces, the conditions imposed on the torques here are more moderate. They are expressed by inequality (10.5).

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